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UNITÉ DE RECHERCHE  
INRIA-ROCQUENCOURT

Institut National  
de Recherche  
en Informatique  
et en Automatique

Domaine de Voluceau  
Rocquencourt  
BP 105  
78153 Le Chesnay Cedex  
France  
Tél (1) 39 63 55 11

## Rapports de Recherche

N° 1209

*Programme 7*  
*Calcul Scientifique,*  
*Logiciels Numériques et Ingénierie Assistée*

### VARIOUS NONCONFORMING APPROXIMATION OF THIN SHELL PROBLEMS AND THE QUESTION OF THEIR CONVERGENCE

Pascal TROUVÉ

Avril 1990





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Pascal Trouvé

Thomson CSF / Central Research Laboratory  
Domaine de Corbeville, 91404 Orsay Cedex, France

and

INRIA, Projet MODULEF,  
Domaine de Voluceau, Rocquencourt, 78153 Le Chesnay, France

## Abstract

Firstly, we recall some nonconforming displacement finite element methods for the approximation of linear thin shells problems. Secondly, we analyze the consistency of a geometrical nonconforming approximation of moderately thick and thin shells. Finally, we discuss a discrete Kirchhoff facet-like finite element method for deep thin shells.

# DIVERSES APPROXIMATIONS NONCONFORMES DE PROBLEMES DE COQUES MINCES ET LA QUESTION DE LEUR CONVERGENCE

## Résumé

Dans ce rapport, nous étudions la consistance des approximations non conformes géométriques de problèmes de coques modérément épaisses ou minces, c'est à dire à l'aide d'éléments finis plats (la surface moyenne de la coque est approchée par un ensemble de facettes triangulaires planes). En premier lieu, nous rappelons quelques résultats sur des espaces d'éléments finis nonconformes, puis nous analysons la consistance d'une telle approximation géométrique pour différents modèles. Enfin, nous discutons une méthode du type Kirchhoff discret facettes planes pour les coques minces profondes.

# 1 INTRODUCTION

For the analysis of shells problems, engineers may use various finite element techniques (conforming or nonconforming displacement methods, with or without shear energy, etc.). In particular, when solving nonlinear problems (with incremental methods, for example) the introduction of an additional approximation, the (conforming or nonconforming) approximation of the geometry becomes very useful. In this framework, the motivations for the use of nonconforming finite element methods are that : it allows the use of low-order interpolations, it is easier to implement (even for curved domains) and it leads generally to good reduction of computation time. In the case of nonconforming displacement methods, one gets sparser matrix, which lower the solving time of the basic linear system when a direct method (such as Choleski's method) is used. In the case of geometrically nonconforming approximations (such as plane facets approximations of the middle surface of the shell), faster are the evaluations of elementary stiffness matrices. In this paper, we give a review of combined approaches, and we discuss convergence properties of the different methods.

## 1.1 Moderately thick to thin shell models

A shell is a **three-dimensional continuum medium**, i.e. a subset  $\bar{C}$  of the Euclidean space  $\mathcal{E}^3$  (provided with an orthonormal reference system  $(0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ ), defined by :

$$\begin{aligned} \bar{C} = \{m \in \mathcal{E}^3, \quad O\vec{m} = \vec{\phi}(\xi^1, \xi^2) + \xi^3 \vec{a}^3, \\ -\frac{1}{2} e(\xi^1, \xi^2) \leq \xi^3 \leq \frac{1}{2} e(\xi^1, \xi^2), \xi = (\xi^1, \xi^2) \in \Omega \subset \mathcal{E}^2\}, \end{aligned}$$

i.e. to each point  $m \in C$  with coordinates  $(x^i)$  with regard to the basis  $(\vec{e}_i)$ , we associate a **curvilinear coordinate system**  $(\xi^1, \xi^2, \xi^3)$  ;  $\Omega$  is a bounded open set in the plane  $\xi^2$ , with boundary  $\Gamma$  ; the **middle surface**  $\bar{S}$  of the shell is defined as the image in  $\mathcal{E}^3$  of  $\bar{\Omega}$  through a **mapping**  $\vec{\phi} : \bar{\Omega} \subset \mathcal{E}^2 \rightarrow \mathcal{E}^3$ , i.e.

$$\bar{S} = \{\vec{\phi}(\xi), \xi = (\xi^1, \xi^2) \in \bar{\Omega}\}.$$

Let us assume that the mapping  $\vec{\phi}$  satisfies :

- (i)  $\vec{\phi} \in (C^3(\bar{\Omega}))^3$  ;
- (ii) any point of the surface  $\bar{S}$  is **regular**, i.e. the vectors  $\vec{a}_\alpha = \frac{\partial \vec{\phi}}{\partial \xi^\alpha} = \vec{\phi}_{,\alpha}$  are linearly independent.

The geometry of the surface  $S$  is then characterized by its three **fundamental forms**  $(a_{\alpha\beta})$ ,  $(b_{\alpha\beta})$ ,  $(c_{\alpha\beta})$ , given by :

- the **metric**  $a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta$ ,  $\det(a_{\alpha\beta}) = a > 0$  ; to each point  $\xi \in \bar{\Omega}$ , we associate to the covariant basis  $(\vec{a}_\alpha(\xi))$  of the tangent plane at the point  $\vec{\phi}(\xi)$  the contravariant basis  $(\vec{a}^\alpha(\xi))$  by using the relations :  $\vec{a}^\alpha \cdot \vec{a}_\beta = \delta^\alpha_\beta$  (Kronecker's symbol) ;

- the **curvatures**  $b_{\alpha\beta} = b_{\beta\alpha} = -\vec{a}_\alpha \cdot \vec{a}_{3,\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta}$  ;

-  $c_{\alpha\beta} = c_{\beta\alpha} = b_\alpha^\beta b_{\lambda\beta} = a^{\lambda\mu} b_{\mu\alpha} b_{\lambda\beta}$ , where  $(a^{\lambda\mu})$  denotes the inverse matrix of  $(a_{\alpha\beta})$  ;

and we determine the Christoffel's coefficient by :  $\Gamma_{\beta\gamma}^\alpha = \Gamma_{\gamma\beta}^\alpha = \vec{a}^\alpha \cdot \vec{a}_{\beta,\gamma}$ , that allows us to introduce the **covariant differentiation** . | (as the basis  $(\vec{a}_i)$ , with  $\vec{a}_3 = \vec{a}^3 = \frac{1}{\sqrt{a}} (\vec{a}_1 \times \vec{a}_2)$ , does not necessarily define an orthonormal basis of vectors), i.e.

$$\begin{cases} T_{\alpha|\beta} &= T_{\alpha,\beta} - \Gamma_{\alpha\beta}^\lambda T_\gamma, \quad T_{3|\alpha} = T_{3,\alpha}, \\ T_{\alpha\beta|\gamma} &= T_{\alpha\beta,\gamma} - \Gamma_{\alpha\gamma}^\lambda T_{\lambda\beta} - \Gamma_{\beta\gamma}^\lambda T_{\alpha\lambda}, \quad \text{etc.} \end{cases}$$

In the following we will consider that the shell  $\bar{\mathcal{C}}$  is clamped on the part  $\partial\mathcal{C}_o = \bar{\phi}(\Gamma_o)$  of its boundary  $\partial\mathcal{C}$ , with  $\text{meas}(\gamma_o) > 0$ , and it is submitted to volumic and surface loads with resultant  $\vec{p} = p^i \vec{a}_i$  on the middle surface. We will assume that such loads only produce **small deformations** of the shell in order that the image  $M$  of a point  $m$  after deformation is given by

$$O\vec{M} = O\vec{m} + \vec{U}, \quad \text{with } \vec{U} = \vec{u}(\xi) + \xi^3 \vec{\delta}(\xi),$$

where  $\vec{u}_i = u_i \vec{a}_i$  is the **displacement** field of the middle surface  $\mathcal{S}$ , and  $\vec{\delta} = \delta_i \vec{a}_i$  is the **rotation** field of the normals to  $\mathcal{S}$ . The **three-dimensional strain tensor**  $(\gamma_{ij}^*)$  is then approximated by

$$(1) \quad \begin{cases} \gamma_{\alpha\beta}^* &= \frac{1}{2} [\mu_\alpha^\lambda (\gamma'_{\lambda\beta} + \xi^3 \kappa'_{\lambda\beta}) + \mu_\beta^\lambda (\gamma'_{\lambda\alpha} + \xi^3 \kappa'_{\lambda\alpha})], \\ \gamma_{\alpha 3}^* &= \frac{1}{2} [\gamma_{\alpha 3} + \xi^3 \kappa_{\alpha 3}], \\ \gamma_{33}^* &= \gamma_{33}, \end{cases}$$

with :  $\mu_\alpha^\lambda = \delta_\alpha^\lambda - \xi^3 b_\alpha^\lambda$ ,

$$(2) \quad \begin{cases} \gamma'_{\alpha\beta} &= u_{\alpha|\beta} - b_{\alpha\beta} u_3, \\ \kappa'_{\alpha\beta} &= \delta_{\alpha|\beta} - b_{\alpha\beta} \delta_3, \\ \gamma_{\alpha 3} &= \delta_\alpha + u_{3|\alpha} + b_\alpha^\lambda u_\lambda, \\ \kappa_{\alpha 3} &= \delta_{3,\alpha}, \\ \gamma_{33} &= \delta_3, \end{cases}$$

and the **three-dimensional stress tensor**  $(\tau^{ij})$  is deduced from the constitutive law

$$(3) \quad \tau^{ij} = C^{ijkl} \gamma_{kl}^*$$

where  $(C^{ijkl})$  is the three-dimensional tensor of elastic moduli.

The basic assumptions introduced by [Naghdi P.M. (1963)] are :  $\delta_3 \equiv 0$  and  $\sigma_{33} \equiv 0$ , i.e.  $\kappa'_{\alpha\beta} = \delta_{\alpha|\beta}$  and  $\gamma_{33}^* = -\frac{C^{\alpha\beta 33}}{C^{3333}} \gamma_{\alpha\beta}^*$  (as  $C^{\alpha 333} = 0$ ), and also

$$\gamma_{\alpha 3}^* = \frac{5}{8} \gamma_{\alpha 3} [1 - (\frac{2\xi^3}{e})^2].$$

Thus the strain energy of the shell is approximated by :

$$(4) \quad J_\delta = \frac{1}{2} \int_\Omega [B_0^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta} \gamma_{\lambda\mu} + 2B_1^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta} \kappa_{\lambda\mu} + B_2^{\alpha\beta\lambda\mu} \kappa_{\alpha\beta} \kappa_{\lambda\mu} + \frac{5}{6} B_0^{\alpha 3\beta 3} \gamma_{\alpha 3} \gamma_{\beta 3}] \sqrt{a} \, d\xi$$

where we have used the expressions

$$(5) \quad \begin{cases} \gamma_{\alpha\beta} &= \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3, \\ \kappa_{\alpha\beta} &= \frac{1}{2} (\delta_{\alpha|\beta} + \delta_{\beta|\alpha}), \end{cases}$$

and

$$\begin{cases} B_n^{\alpha\beta\lambda\mu} &= \int_{-e/2}^{e/2} A^{\kappa\beta\nu\mu} \mu_\kappa^\alpha \mu_\nu^\lambda \mu(\xi^3)^n d\xi^3, \\ B_0^{\alpha 3\beta 3} &= \frac{15}{8} \int_{-e/2}^{e/2} C^{\alpha 3\beta 3} [1 - (\frac{2\xi^3}{e^2})^2] \mu d\xi^3, \end{cases}$$

with  $\mu = \det(\mu_\beta^\alpha)$  and  $A^{\alpha\beta\lambda\mu} = C^{\alpha\beta\lambda\mu} - \frac{C^{\alpha\beta 33}}{C^{3333}} C^{33\lambda\mu}$ . Therefore, if one assumes the medium to be homogeneous in the thickness of the shell, one gets in the case of **moderately thick shells**, i.e. when  $(\frac{e}{R})^2 \ll 1$  (where  $R$  is the minimum absolute value of radii of curvatures) :

$$\begin{cases} B_0^{\alpha\beta\lambda\mu} &= e E^{\alpha\beta\lambda\mu}, \\ B_1^{\alpha\beta\lambda\mu} &= \frac{e^3}{12} E^{\alpha\epsilon\lambda\eta} (\delta_\epsilon^\beta b_\eta^\mu + \delta_\eta^\mu b_\epsilon^\beta - b_\nu^\nu \delta_\epsilon^\beta \delta_\eta^\mu), \\ B_2^{\alpha\beta\lambda\mu} &= \frac{e^3}{12} E^{\alpha\beta\lambda\mu}, \\ B_0^{\alpha 3\beta 3} &= e E^{\alpha 3\beta 3}, \end{cases}$$

where, for elastic homogeneous and isotropic media, one has

$$\begin{cases} E^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)} (a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu}), \\ E^{\alpha 3\beta 3} &= \frac{E}{2(1+\nu)} a^{\alpha\beta}. \end{cases}$$

Next, by noticing that

$$|B_1^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta} \kappa_{\lambda\mu}| \leq C(\frac{e}{R}) [B_0^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta} \gamma_{\lambda\mu}]^{1/2} [B_2^{\alpha\beta\lambda\mu} \kappa_{\alpha\beta} \kappa_{\lambda\mu}]^{1/2},$$

we obtain a first-order approximation of (4) for  $(\frac{e}{R})$  **sufficiently small**, i.e.

$$(6) \quad J_S \sim \frac{1}{2} \int_\Omega \{ e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta} \gamma_{\lambda\mu} + \frac{e^2}{12} \kappa_{\alpha\beta} \kappa_{\lambda\mu}] + \frac{5}{6} e E^{\alpha 3\beta 3} \gamma_{\alpha 3} \gamma_{\beta 3} \} \sqrt{a} d\xi.$$

In the following we consider that the expression (6) of the strain energy applies to **moderately curved shells**.

A second approach developped by [Naghdi P.M. (1972)] amounts to consider first-order expansions in (1), and mainly

$$(7) \quad \gamma_{\alpha\beta}^* \sim \gamma_{\alpha\beta} + \xi^3 \chi_{\alpha\beta}$$

where

$$(8) \quad \chi_{\alpha\beta} = \frac{1}{2} (\delta_{\alpha|\beta} + \delta_{\beta|\alpha}) - \frac{1}{2} (b_\beta^\lambda u_{\lambda|\alpha} + b_\alpha^\lambda u_{\lambda|\beta}) + c_{\alpha\beta} u_3.$$

Then, under similar assumptions, we derive a new approximation of (4) for  $(\frac{e}{R}) \ll 1$  :

$$(9) \quad J_S \sim \frac{1}{2} \int_{\Omega} \{ e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta} \gamma_{\lambda\mu} + \frac{e^2}{12} \chi_{\alpha\beta} \chi_{\lambda\mu}] + \frac{5}{6} e E^{\alpha 3 \beta 3} \gamma_{\alpha 3} \gamma_{\beta 3} \} \sqrt{a} d\xi.$$

that stands for the strain energy of **deep** (moderately thick) shells.

Now, if one completes this last model with the plane state of stress in the shell (i.e.  $\sigma_{i3} \equiv 0$ ), one finds [Koiter W.T. (1966)] linear model of **thin** shells, i.e.

$$(10) \quad J_S \sim \frac{1}{2} \int_{\Omega} e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta} \gamma_{\lambda\mu} + \frac{e^2}{12} \bar{\rho}_{\alpha\beta} \bar{\rho}_{\lambda\mu}] \sqrt{a} d\xi.$$

where the change of curvature tensor  $\bar{\rho}_{\alpha\beta}$  is given by

$$(11) \quad \bar{\rho}_{\alpha\beta} = u_{3|\alpha\beta} - c_{\alpha\beta} u_3 + b_{\alpha}^{\lambda} u_{\lambda|\beta} + b_{\beta}^{\lambda} u_{\lambda|\alpha} + b_{\alpha|\beta}^{\lambda} u_{\lambda}.$$

Let us add that, in the thin shell approximation, the first model giving (6) does not produce the shallow shell model which for  $\bar{\rho}_{\alpha\beta}$  is replaced by  $u_{3|\alpha\beta}$  in (10), but it leads to replace  $\bar{\rho}_{\alpha\beta}$  by

$$(12) \quad \tilde{\rho}_{\alpha\beta} = u_{3|\alpha\beta} + \frac{1}{2} (b_{\alpha}^{\lambda} u_{\lambda|\beta} + b_{\beta}^{\lambda} u_{\lambda|\alpha}) + b_{\alpha|\beta}^{\lambda} u_{\lambda}.$$

## 2 NONCONFORMING DISPLACEMENT METHODS

In this section, we describe the approximation of linear Koiter's model of thin shell problems through **nonconforming finite element spaces** (cf. [Trouvé P. (1990)]). Let us denote  $\vec{u}$  the unique solution (see [Bernadou M., Ciarlet P.G. (1976)]) of the continuous problem

$$(13) \quad a(\vec{u}, \vec{v}) = \ell(\vec{v}), \forall \vec{v} \in \vec{V},$$

with

$$\begin{cases} \vec{V} &= \{ \vec{v} \in (H_1(\Omega))^2 \times H^2(\Omega) ; \vec{v}|_{\Gamma_o} = \vec{0}, \partial_n v_3|_{\Gamma_o} \} \\ a(\vec{u}, \vec{v}) &= \int_{\Omega} e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{u}) \gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12} \bar{\rho}_{\alpha\beta}(\vec{u}) \bar{\rho}_{\alpha\beta}(\vec{v})] \sqrt{a} d\xi \\ \ell(\vec{v}) &= \int_{\Omega} \vec{p} \cdot \vec{v} \sqrt{a} d\xi \end{cases}$$

and expressions (5) and (11). We only consider the effect of distributed vector loads  $\vec{p}$  on  $\mathcal{S}$ , and we assume homogeneous boundary conditions on  $\Gamma_1 = \partial\Omega/\Gamma_o$ , for the sake of clarity. If the domain  $\Omega$  is sufficiently regular, and if the coefficients involved in (13) are sufficiently smooth, one can conjecture that  $\vec{u} \in (H^3(\Omega))^2 \times H^4(\Omega)$ . Therefore, there exists two linear continuous operators  $A$  and  $B$ , from  $(H^3(\Omega))^2 \times H^4$  into  $(L^2(\Omega))^3$  and  $(L^2(\Gamma_1))^4$ , such that

$$a(\vec{u}, \vec{v}) = \langle A\vec{u}, \vec{v} \rangle + \langle B\vec{u}, \vec{v} \rangle, \quad \forall \vec{v} \in \vec{V},$$

where  $\langle A\vec{u}, \vec{v} \rangle = \langle \vec{p}, \vec{v} \rangle$  expresses the equilibrium equations, and  $\langle B\vec{u}, \vec{v} \rangle = 0$  are homogeneous boundary conditions of the Neumann type on  $\Gamma_1$ .

In the following we use a finite element partition of  $\Omega$ , i.e.  $\Omega_h = \bigcup_{K \in \mathcal{T}_h} K$  where  $\mathcal{T}_h$  is a (uniformly) regular triangulation in the sense of [Ciarlet P.G. (1978)]. The nonconforming displacement methods consist in solving the discrete problem : find  $\vec{u}_h \in \vec{V}_h$  such that

$$(14) \quad a_h(\vec{u}_h, \vec{v}_h) = \ell(\vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h,$$

with

$$\vec{V}_h = \{ \vec{v}_h \in (H^1(\Omega_h))^2 \times H^2(\Omega_h) ; \vec{v}_h|_K \in \vec{P}_K, \forall K \in \Omega_h ; \lambda(\vec{v}_h)|_{\Gamma_o} = 0, \forall \lambda \in \Sigma_h \}$$

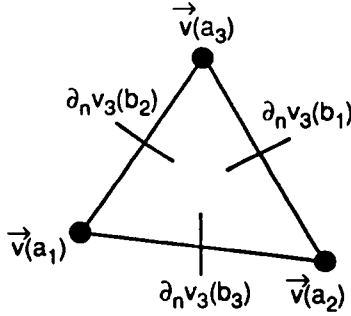
where  $\vec{P}_K$  is a space of polynomials, and  $\Sigma_h$  is the set of degrees of freedom — from a practical point of view, the tangential components of the displacements are approximated by conforming  $C^0$ -finite elements whereas the transverse component may be approximated by finite elements that may not even be  $C^0(\bar{\Omega})$ , the elements of  $\Sigma_h$  are the values of the displacements at each vertex of  $\mathcal{T}_h$ , and the values (or mean values) of the first derivatives of the transverses displacements on the edges, i.e. on  $\partial\Omega_h$  - ,

$$a_h(\vec{u}_h, \vec{v}_h) = \langle A\vec{u}_h, \vec{v}_h \rangle_h + \langle B\vec{u}_h, \vec{v}_h \rangle_h,$$

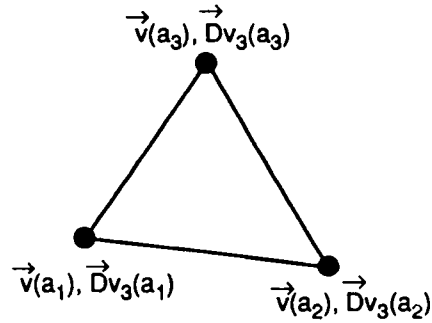


where each bilinear form, previously set over  $\Omega$ , is now defined on  $\Omega_h$ , which justifies the use of a subscript  $h$  (i.e. the integrals over  $\Omega$  are replaced by sums of integrals over each  $K \in \mathcal{T}_h$ ).

In [Trouvé P. (1990)], we have given sufficient conditions on the definition of finite element spaces to obtain convergence of the nonconforming methods, and we have studied well-known families of low-order **semiloof** or  $C^0$ -finite elements. The first example is the Morley shell element for which  $\vec{P}_K = (P_1(K))^2 \times P_2(K)$ , with  $\Sigma_h = \bigcup_{K \in \mathcal{T}_h} \Sigma_K$  and  $\Sigma_K = \{\vec{v}(a), \forall a \text{ vertex of } K; \partial_n v_3(b), \forall b \text{ midside point of } K\}$  ( $K$  being a triangle). A second example is the  $C^0$ -DKT shell finite element (see [Bernadou M., Mato Eiroa P., Trouvé P. (1990)]) where  $\vec{P}_K = (P_2(K))^2 \times P'_3(K)$  and  $\Sigma_K = \{\vec{v}(a), \vec{D}v_3(a), \forall a \text{ vertex of } K\}$  ( $K$  is a triangle). Besides the consistent approximation of  $a_h(.,.)$ , the basic idea of this approach is to replace the first derivatives of  $v_3$  by discrete ones in the **nodal scheme** :  $\partial_t v_3$  is replaced by  $\partial_t^h v_3 \in P_2(S)$  and  $\partial_n v_3$  by  $\partial_n^h v_3$  in  $P_1(S)$  along each side  $S \subset \partial K$ , this allows to control complete continuity of the displacements and their discrete derivative through interfaces between elements.



The Morley shell element



The  $C^0$ -DKT shell element

The numerical analysis of both methods leads to the following error estimate

$$(15) \quad \|\vec{u} - \vec{u}_h\|_h \leq Ch \{ \|u_1\|_{3,\Omega}^2 + \|u_2\|_{3,\Omega}^2 + \|u_3\|_{4,\Omega}^2 \}^{1/2}$$

where

$$(16) \quad \|\vec{v}_h\|_h = \{ \|v_{h1}\|_{1,\Omega}^2 + \|v_{h2}\|_{1,\Omega}^2 + \sum_{K \in \Omega_h} \|v_{h3}\|_{2,K}^2 \}^{1/2}.$$

The rate of convergence deduced from (15) may be obtained through lower regularity assumptions (see [Trouvé P. (1990)] and [Bernadou M., Mato Eiroa P., Trouvé P. (1990)] for technical details). Moreover, in the case of  $C^0$ -DKT approach, this results may be raised to an  $O(h^2)$ -rate of convergence in some particular cases (as in plates problems).

**Remark 2.1 :** *It was shown by [Arnold D.N., Brezzi F. (1985)] that a slight modification of the Morley method is equivalent to the Hellan-Herrmann-Johnson mixed-hybrid finite element method, but the first method avoids the use of post-processing techniques to recover displacements, while both methods leads to the same error estimates. It is also possible to describe the Morley method as being equivalent to a primal hybrid method (see [Brezzi F. (1975)] [Brezzi F. (1977)] and [Brezzi F., Marini L.D. (1975)], [Kikuchi F., Ando Y. (1972a)], [Kikuchi F., Ando Y. (1972b)]) :*

$$a_h(\vec{u}_h, \vec{v}_h) + b_h(\vec{M}_h, \vec{v}_h) = \ell(\vec{v}_h), \quad \forall \vec{v}_h \in \vec{X}_h,$$

$$b_h(\vec{N}_h, \vec{u}_h) = 0, \quad \forall \vec{N}_h \in \vec{Y}_h,$$

where

$$b_h(\vec{N}_h, \vec{v}_h) = \sum_{S \in \partial\Omega_h} \int_S \vec{N}_h \cdot \vec{D}v_h \, ds;$$

$$\vec{X}_h = \{\vec{v}_h \in (H^1(\Omega_h))^2 \times H^2(\Omega_h); \vec{v}_h|_K \in (P_1(K))^2 \times P_2(K); \forall K \in \Omega_h\},$$

$$\vec{Y}_h = \{\vec{N}_h \in (H^{-1/2}(\partial\Omega_h))^2; \vec{N}_h|_S \in (P_0(S))^2, \forall S \in \partial\Omega_h; \vec{N}_h|_{\Gamma_1} = \vec{0}\}.$$

Concerning the discrete Kirchhoff finite element method, it may be rewritten as a primal mixed method (see for instance [Destuynder P., Nedelec J.C. (1986)], [Kikuchi F. (1983)], [Pitkaranta J. (1988)]) :

$$a_h[(\vec{v}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] + b_h[\vec{p}_h, (\vec{v}_h, \vec{\beta}_h)] = \ell(\vec{v}_h), \quad \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h,$$

$$b_h[\vec{q}_h, (\vec{u}_h, \vec{\delta}_h)] = 0, \quad \forall \vec{q}_h \in \vec{Q}_h,$$

where we have denoted

$$a_h[(\vec{v}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] = \int_{\Omega_h} e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{v}_h) \gamma_{\lambda\mu}(\vec{v}_h) + \frac{e^2}{12} \chi_{\alpha\beta}(\vec{v}_h, \vec{\delta}_h) \chi_{\lambda\mu}(\vec{v}_h, \vec{\beta}_h)] \sqrt{a} \, d\xi,$$

$$b_h[\vec{q}_h, (\vec{v}_h, \vec{\beta}_h)] = \int_{\Omega_h} q_h^\alpha (\beta_{h\alpha} + v_{h3,\alpha} + b_\alpha^\lambda v_{h\lambda}) \sqrt{a} \, d\xi,$$

$$\vec{W}_h = \{(\vec{v}_h, \vec{\beta}_h) \in (H^1(\Omega))^5; \quad (\vec{v}_h, \vec{\beta}_h)|_K \in (P_2(K))^2 \times P_3'(K) \times (P_2(K))^2, \\ \forall K \in \Omega_h; \vec{v}_h|_{\Gamma_o} = \vec{\beta}_h|_{\Gamma_o} = \vec{0}\},$$

$$\vec{Q}_h = \{\vec{q}_h \in (H^{-1}(\Omega_h))^2, \quad q_{h,\alpha}^\alpha \in H^{-1}(\Omega_h); \vec{q}_h|_K \in (P_2(K))^2, \\ \text{and } q_h^\alpha n_\alpha = 0 \text{ at midside points of } K, \forall K \in \Omega_h\},$$

(and this method gives an  $(h^2)$ -rate of convergence in the case of plates problem). ■

Following [Ruas V. (1978)], no additional approximations is required to preserve the  $O(h)$ -rate of convergence of the Morley method in the case of curved boundaries  $\partial\Omega$  (when  $\Omega_h$  remains polygonal). Next, one easily extends results obtained by [Ciarlet P.G. (1978)] on the conforming approximation of the geometry of the shell to nonconforming displacement

finite element methods. This amounts to use interpolation of the mapping  $\vec{\phi}$  by polynomials of the third degree (at least) elementwise. Moreover one can take into account the effects of numerical integration by using similar error estimates as in [Bernadou M. (1980)]. Thus, a one point quadrature formula is sufficient to get an  $O(h)$ -rate of convergence in the Morley finite element methods, whereas a three-points quadrature formula seems to be sufficient in the  $C^0$ -discrete Kirchhoff finite element methods. In the following paragraph, we study a non-conforming approximation of the geometry using triangular plane facet elements associated to the Morley method, then to the DKT-method.

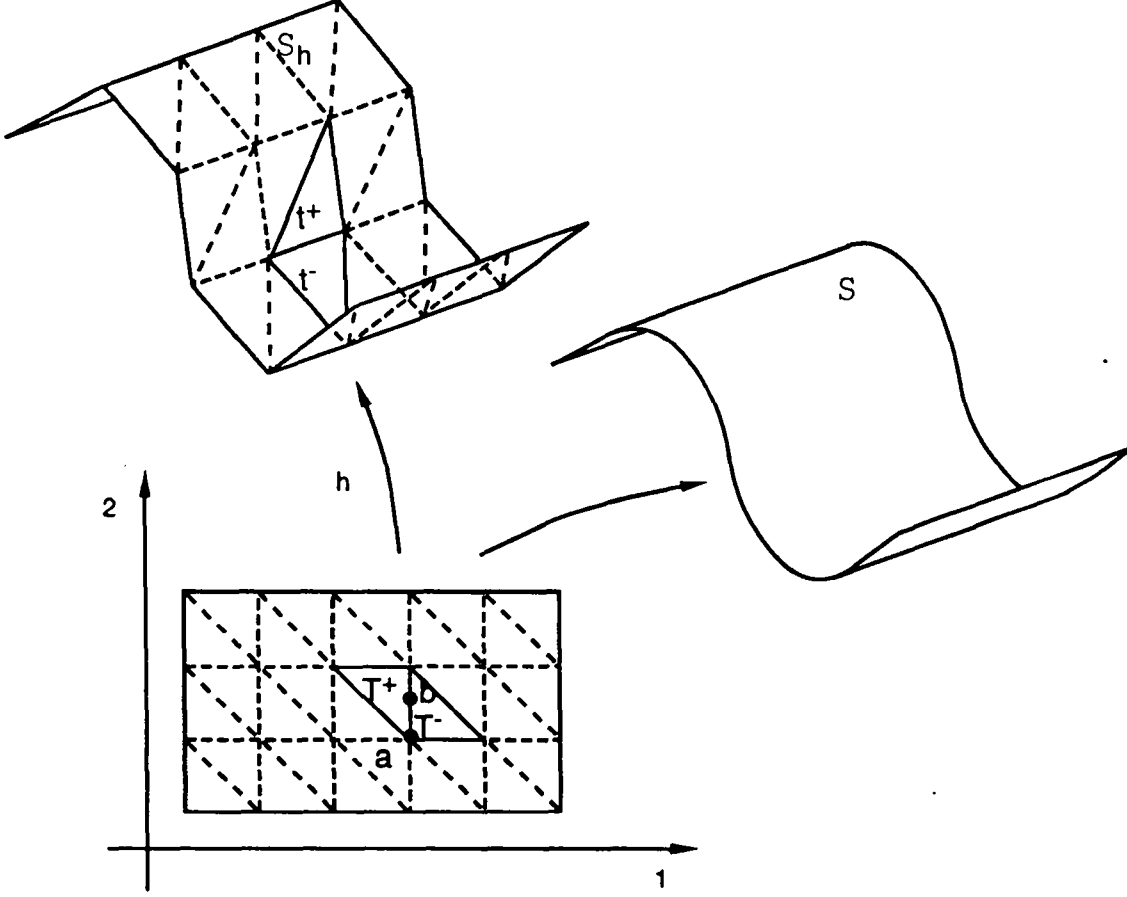
### 3 GEOMERICALLY NONCONFORMING FINITE - ELEMENT METHODS

It is well-known now that the approximation of general arches problems by straight beams converges (see [Bernadou M., Ducatel Y. (1982)] and the bibliography therein). Then, for the approximation of circular cylindrical shells by flat plate elements, the use of conforming displacement methods does not converge to the solution of the continuous thin shell problems (see [Bernadou M., Ducatel Y., Trouvé P. (1988)]), whereas the use of nonconforming displacement methods may converges to the right solution (see [Kikuchi F. (1984)]). Concerning the approximation of general shells problems by flat plate elements combined to conforming displacement spaces, it was proved by [Bernadou M., Trouvé P., Ducatel Y. (1989)] that pseudo-convergence is obtained for sufficiently shallow shells while extraneous bending effects appears for deep thin shells in regions where the membrane state of stress predominates, and this results seems to be valid whatever the thin shell model may be, whatever the conforming finite elements are, or whatever the kind of compatibility relations used. The purpose of this paragraph is to conclude about the combined use of nonconforming finite element spaces and flat plate approximations of general shells. Let us notice that the semi-loof flat plate finite element method proposed by [Dawe D.J. (1972)] allows a pointwise description of the middle surface without any need for introducing drilling degrees of freedom (as in [Bernadou M., Trouvé P. (1990)], [Hughes T.J.R., Brezzi F. (1988)]).

#### 3.1 Analysis of the Morley-Dawe method (see [Trouvé P. (1988)])

The approximation of the mapping  $\vec{\phi} = \phi^i \vec{e}_i$  consists in approaching each component  $\phi^i$  by its  $P^1$ -interpolate elementwise, which is uniquely determined by its values at the vertices of each triangle  $K \in \Omega_h$ . It amounts to replace the continuous middle surface  $\bar{S}$  by a faceted-like surface  $\bar{S}_h$ . By construction the images of the vertices of  $\mathcal{T}_h$  in the reference domain  $\Omega_h$  by the mapping  $\vec{\phi}_h$  belongs to the continuous middle-surface  $\bar{S}$  (see figure). Then to each flat triangle  $K_h = \vec{\phi}_h(K)$ , we associate a local basis :  $\vec{a}_{h\alpha} = \vec{\phi}_{h,\alpha}$ ,  $\sqrt{a_h} = |\vec{a}_{h1} \times \vec{a}_{h2}|$ ,  $\vec{a}_{h3} = \frac{1}{\sqrt{a_h}} (\vec{a}_{h1} \times \vec{a}_{h2}) = \vec{a}_h^3$ ,  $a_{h\alpha\beta} = \vec{a}_{h\alpha} \cdot \vec{a}_{h\beta}$ ,  $\vec{a}_h^\alpha = a_h^{\alpha\beta} \vec{a}_{h\beta}$ , etc, and  $b_{h\alpha\beta} = b_{h\alpha}^\beta = c_{h\alpha\beta} = \Gamma_{h\alpha\beta}^\gamma = 0$ . Therefore, usual geometrical coefficients will be constants or null on each triangle, and they may present some discontinuities from a triangle to another, due to the discontinuities of the first derivatives of  $\vec{\phi}_h$ . Next the displacements  $\vec{v}_h$  are approximated, triangle by triangle, by the Morley shell elements, which allow to evaluate on each triangle

$$(17) \quad \begin{cases} \tilde{\gamma}_{h\alpha\beta}(\vec{v}_h) &= \frac{1}{2} (\tilde{v}_{h\alpha,\beta} + \tilde{v}_{h\beta,\alpha}), \\ \tilde{\rho}_{h\alpha\beta}(\vec{v}_h) &= \tilde{v}_{h3,\alpha\beta}. \end{cases}$$



So we get two types of discontinuity at the interface between two triangle of  $\Omega_h$  : a geometrical discontinuity (of the local basis  $\vec{a}_{hi}$  and the metric  $a_{h\alpha\beta}$ ), and the discontinuity of the transverse displacement  $\tilde{v}_{h3}$ , and this results (a priori) in a complete discontinuity of  $\vec{\tilde{v}}_h$ . Now, to define the discrete space  $\tilde{\tilde{V}}_h$ , we introduce some **compatibility relations** between degrees of freedom of two adjacent triangles. They are chosen in order to preserve (at least) the rigid body movement from a triangle to another, which leads to a well-posed discrete problem. Here, we set

- (i) the continuity of the displacement vector  $\vec{\tilde{v}}_h$  at each vertex of the surface  $S$ , or equivalently at each vertex  $a$  of  $\mathcal{T}_h$  ;

$$(18) \quad \vec{\tilde{v}}_h(a^+) = \vec{\tilde{v}}_h(a^-), \forall a^+ = a^- = a \text{ vertex of } \mathcal{T}_h,$$

(ii) the continuity of the rotation around the common edge at its midside point  
i.e.

$$(19) \quad \partial_n \tilde{v}_{h3}(b^+) + \partial_{n^-} \tilde{v}_{h3}(b^-) = 0, \quad \forall b^+ = b^- = b \text{ midside node in } \mathcal{T}_h,$$

where  $\vec{n}^+$  and  $\vec{n}^-$  denotes the unit exterior normal to the common edge of adjacent triangles  $K^+$  and  $K^-$  (i.e.  $\vec{n}^- = -\vec{n}^+$ ). Then we define

$$\left\{ \begin{array}{l} \vec{V}_h = \{ \vec{v}_h \in (H^1(\Omega_h))^2 \times H^2(\Omega_h) ; \vec{v}_h|_K \in \vec{P}_K ; \\ \vec{v}_h \text{ satisfies (18) and (19) ; } \vec{v}_h(a) = \vec{0}, \forall a \in \Gamma_o, \partial_n \tilde{v}_{h3}(b) = 0, \forall b \in \Gamma_o \}, \end{array} \right.$$

and we provide it with the discrete norm (16) (taken over  $\Omega_h$ ). In fact, the compatibility relations allow the existence of a mapping  $\mathcal{F}_h$ , one-to-one and onto, between the discrete spaces  $\vec{V}_h$  and  $\tilde{V}_h$ , i.e. the relations

$$\left\{ \begin{array}{l} \vec{v}_h(a) = \tilde{v}_h(a), \quad \forall a \text{ vertex of } \mathcal{T}_h, \\ \partial_n \tilde{v}_{h3}(b) = \frac{e^{\lambda\mu}}{\sqrt{a(b)}} [v_{h3,\mu}(b) + b_\mu^\nu(b) v_{h\nu}(b)] (\vec{a}_\lambda(b) \cdot \vec{t}_h(b)), \quad \forall b \text{ midside node of } \mathcal{T}_h, \end{array} \right.$$

where  $\vec{t}_h$  is the image through  $\phi_h$  of the unit tangent vector to the edge of  $\Omega_h$  with midside node  $b$  (and  $e^{\lambda\mu} = 0$  if  $\lambda = \mu$ ,  $e^{12} = 1$ ,  $e^{21} = -1$ ), define the bijection  $\mathcal{F}_h$  such that  $\tilde{V}_h = \mathcal{F}_h(\vec{V}_h)$ . This will enable us to make the numerical analysis of the method within the framework of the space  $\tilde{V}_h$ .

The discrete problem associated with the faceted middle surface  $\mathcal{S}_h$  is : Find  $\vec{u}_h \in \vec{V}_h$ , such that

$$(20) \quad \tilde{a}_h(\vec{u}_h, \vec{v}_h) = \tilde{\ell}_h(\vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h,$$

with

$$\begin{aligned} \tilde{a}_h(\vec{u}_h, \vec{v}_h) &= \int_{\Omega_h} e E_h^{\alpha\beta\lambda\mu} [\tilde{\gamma}_{h\alpha\beta}(\vec{u}_h) \tilde{\gamma}_{h\lambda\mu}(\vec{v}_h) + \frac{e^2}{12} \tilde{\rho}_{h\alpha\beta}(\vec{u}_h) \tilde{\rho}_{h\lambda\mu}(\vec{v}_h)] \sqrt{a_h} d\xi, \\ \tilde{\ell}_h(\vec{v}_h) &= \int_{\Omega_h} \vec{p} \cdot \vec{v}_h \sqrt{a_h} d\xi. \end{aligned}$$

By using the mapping  $\mathcal{F}_h$ , this problem can be rewritten as : find  $\vec{u}_h \in \vec{V}_h$ , such that :

$$(21) \quad \hat{a}_h(\vec{u}_h, \vec{v}_h) = \hat{\ell}_h(\vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h,$$

where we use the following correspondences

$$\hat{a}_h(\vec{v}_h, \vec{w}_h) = \tilde{a}_h(\vec{v}_h, \vec{w}_h), \quad \forall \vec{v}_h = \mathcal{F}_h(\vec{v}_h), \vec{w}_h = \mathcal{F}_h(\vec{w}_h), \vec{v}_h, \vec{w}_h \in \vec{V}_h,$$

$$\hat{\ell}_h(\vec{v}_h) = \tilde{\ell}_h(\vec{v}_h), \quad \forall \vec{v}_h = \mathcal{F}_h(\vec{v}_h), \vec{v}_h \in \vec{V}_h.$$

Therefore (21) is a new approximation to problem (13).

Following Strang's lemma and interpolation theory in Sobolev spaces (see [Ciarlet P.G. (1978)]), the convergence of the present method amounts to the consistency of the approximations  $|a_h(\vec{v}_h, \vec{w}_h) - \hat{a}_h(\vec{v}_h, \vec{w}_h)|$  and  $|\ell(\vec{v}_h) - \hat{\ell}_h(\vec{v})|$ , which reduces to the estimates of  $|\gamma_{\alpha\beta}(\vec{v}_h) - \tilde{\gamma}_{h\alpha\beta}(\vec{v}_h)|_{0,K}$  and  $|\bar{\rho}_{\alpha\beta}(\vec{v}_h) - \tilde{\rho}_{h\alpha\beta}(\vec{v}_h)|_{0,K}$ . The first error estimate follows from [Bernadou M., Trouvé P., Ducatel Y. (1989)], i.e.

$$(22) \quad |\gamma_{\alpha\beta}(\vec{v}_h) - \tilde{\gamma}_{h\alpha\beta}(\vec{v}_h)|_{0,K} \leq Ch \|\vec{v}_h\|_{1,K}, \quad \forall \vec{v}_h = \mathcal{F}_h(\vec{v}), \quad \vec{v}_h \in \vec{V}_h.$$

Next we get (see [Trouvé P. (1988)] (V proposition 5.4.2)) on each  $K \in \Omega_h$  :

$$(23) \quad \begin{cases} \tilde{\rho}_{h\alpha\beta}(\vec{v}_h) = \bar{\rho}_{\alpha\beta}(\vec{v}_h) + b_{\epsilon\eta}[a^{\lambda\mu}v_{h\mu|\nu} - b_\nu^\lambda v_{h3}] \\ \quad \times \sum_{i=1}^3 A_\lambda^{\epsilon\eta}(b_i)[\xi_i^\nu - \frac{1}{2}(1-\eta_i)\xi_{i+1}^\nu - \frac{1}{2}(1+\eta_i)\xi_{i-1}^\nu](p_{1,i}^\perp)_{,\alpha\beta} \\ \quad + O(h)(\|v_{h1}\|_{1,\infty,K} + \|v_{h2}\|_{1,\infty,K} + \|v_{h3}\|_{2,\infty,K}), \end{cases}$$

where  $b_i, i = 1, 2, 3$ , are the midside nodes of  $K$ ,

$$A_\lambda^{\epsilon\eta}(\xi) = \frac{1}{2} \sum_{k=1}^3 \frac{\partial \lambda_k}{\partial \xi^\lambda} (\xi_k^\epsilon - \xi^\epsilon)(\xi_k^\eta - \xi^\eta), \quad \xi \in K,$$

( $\lambda_k$  being the barycentric coordinates of  $K$ ),  $\eta_i$  are the excentricity parameters of  $K$ , and where  $p_{1,i}^\perp$  is the basis function of the Morley triangle associated with the degree of freedom  $D\tilde{v}_{h3}(b_i)(a_i - c_i)$  (i.e.  $p_{1,i}^\perp(\lambda) = -\frac{1}{|a_i - c_i|} \lambda_i(1 - \lambda_i)$ ,  $a_i - c_i = (\xi_i^\nu - \frac{1}{2}(1 - \eta_i)\xi_{i+1}^\nu - \frac{1}{2}(1 + \eta_i)\xi_{i-1}^\nu)_{\nu=1,2}$ ).

Therefore, the conclusions are similar as in [Bernadou M., Trouvé P., Ducatel Y. (1989)]: the residual term in (23) does not reduce to some  $O(h)$  estimate, and it may be very small in the case of quasi-shallow shells (i.e.  $|b_{\epsilon\eta}| \leq \epsilon$ ,  $\epsilon$  very small) leading to some pseudo-convergence. But, for general deep thin shells case, one has to add a corrector term to  $\tilde{\rho}_{h\alpha\beta}$  elementwise, i.e.

$$(24) \quad \begin{cases} \tilde{\rho}_{h\alpha\beta}^*(\vec{v}_h) = \tilde{v}_{h3,\alpha\beta} \\ \quad - \sum_{i=1}^3 A_\lambda^{\epsilon\eta}(b_i)[\xi_i^\nu - \frac{1}{2}(1-\eta_i)\xi_{i+1}^\nu - \frac{1}{2}(1+\eta_i)\xi_{i-1}^\nu](p_{1,i}^\perp)_{,\alpha\beta} b_{h\epsilon\eta} a_h^{\lambda\mu} \tilde{v}_{h\mu,\nu} \end{cases}$$

where  $b_{h\epsilon\eta}$  is any approximation of the curvatures  $b_{\epsilon\eta}$  satisfying

$$(25) \quad |b_{\epsilon\eta} - b_{h\epsilon\eta}|_{0,\infty,K} = O(h),$$

to obtain the consistency error

$$(26) \quad \begin{cases} |\tilde{\rho}_{\alpha\beta}(\vec{v}_h) - \tilde{\rho}_{h\alpha\beta}^*(\vec{v}_h)|_{0,K} \leq Ch(\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{2,K}^2)^{1/2}, \\ \quad \forall \vec{v}_h = \mathcal{F}_h(\vec{v}), \quad \vec{v}_h \in \vec{V}_h. \end{cases}$$

Afterwards, by replacing  $\tilde{\rho}_{h\alpha\beta}$  by  $\tilde{\rho}_{h\alpha\beta}^*$ , we derive a new finite element method which is convergent for general shells (and sufficiently small  $h$  in order to derive uniform  $\vec{V}_h$ -ellipticity of the discrete bilinear form  $\hat{a}_h(.,.)$  from consistency error estimates), i.e.

$$\|\vec{u} - \vec{\tilde{u}}\|_h \leq (\|\vec{u}_1\|_{3,\Omega}^2 + \|\vec{u}_2\|_{3,\Omega}^2 + \|\vec{u}_3\|_{4,\Omega}^2)^{1/2}.$$

**Remark 3.1 :** *One way to construct approximations of the curvatures  $b_{\alpha\beta}$  on each triangle  $K \in \mathcal{T}_h$  is to consider a patch of four triangles and to construct an isoparametric interpolation of the second order  $\pi\vec{\phi}$  from the values  $\vec{\phi}(a)$ ,  $a$  denotes the vertices of triangles in the patch. Thus*

$$b_{h\alpha\beta|K} = \vec{a}_{h3} \cdot (\pi\vec{\phi})_{,\alpha\beta|K}$$

*satisfies (25). A second approach (from [Nedelec J.C. (1976)]) consists in the approximation of the normals : by adding more data on the description of the continuous middle-surface, for example the normal vector  $\vec{a}_3$  at each vertex of  $\mathcal{T}_h$ , we can define elementwise  $\vec{a}_{h3}$  as being the  $P_1$ -interpolation of  $\vec{a}_3$ , and thus*

$$b_{h\alpha\beta|K} = -\vec{\phi}_{h,\alpha} \cdot \vec{a}_{h3,\beta},$$

*satisfies (25).*

■

### 3.2 Consistency of a flat plate finite element approximation of moderately curved and moderately thick shells

In this paragraph we plan to approximate Naghdi's (1963) model, i.e. find  $(\vec{u}, \vec{\delta}) \in \vec{W} = \{(\vec{v}, \vec{\beta}) \in (H^1(\Omega))^5 ; \vec{v}|_{\Gamma_0} = \vec{\beta}|_{\Gamma_0} = \vec{0}\}$  such that

$$(27) \quad a[(\vec{u}, \vec{\delta}), (\vec{v}, \vec{\beta})] = \ell(\vec{v}), \quad \forall (\vec{v}, \vec{\beta}) \in \vec{W},$$

where the bilinear form  $a[.,.]$  is defined by

$$\begin{aligned} a[(\vec{u}, \vec{\delta}), (\vec{v}, \vec{\beta})] &= \int_{\Omega} \{eE^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{u}) \gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12} \kappa_{\alpha\beta}(\vec{\delta}) \kappa_{\lambda\mu}(\vec{\beta})] \\ &\quad + \frac{5}{6} eE^{\alpha 3 \beta 3} \gamma_{\alpha 3}(\vec{u}, \vec{\delta}) \gamma_{\beta 3}(\vec{v}, \vec{\beta})\} \sqrt{a} d\xi. \end{aligned}$$

Let us assume that this continuous problem admits a unique solution. Here, we choose conforming finite elements of type (1) for the displacements and the rotations. It was shown by [Coutis N. (1980)] that a conforming approximation of the geometry of the shell would consist in polynomial interpolation of the mapping  $\vec{\phi}$  of degree 2 (at least). We consider now a nonconforming approximation of the mapping  $\vec{\phi}$  by using its  $(P_1)^3$ -interpolate  $\vec{\phi}_h$ , which



amounts to replace the continuous middle surface  $\bar{S}$  by an assemblage of triangular plane facets. To the displacement field  $\vec{v}_h$  and rotation field  $\vec{\beta}_h$ , we can associate on each triangle, the plate approximations of strains, i.e.  $\tilde{\gamma}_{h\alpha\beta}(\vec{\beta}_h)$ ,

$$(28) \quad \begin{cases} \tilde{\kappa}_{h\alpha\beta}(\vec{\beta}_h) &= \frac{1}{2} (\tilde{\beta}_{h\alpha,\beta} + \tilde{\beta}_{h\beta,\alpha}) = \tilde{\gamma}_{h\alpha\beta}(\vec{\beta}_h), \\ \tilde{\gamma}_{h\alpha 3}(\vec{v}_h, \vec{\beta}_h) &= \tilde{\beta}_{h\alpha} + \tilde{v}_{h3,\alpha}. \end{cases}$$

For practical reasons (i.e. a pointwise description of the middle-surface of the shell), we introduce a **sixth degree of freedom per node** by perturbing the strain energy of the shell (see [Hughes T.J.R., Brezzi F. (1988)], [Bernadou M., Trouné P. (1990)]). Here this will be done by adding some  $H^1(\Omega_h)$ -bilinear form of the tranverse component of the rotation vector to the discrete bilinear form directly derived from (27). Thus we can set compatibility relations on vectors, which are easy to implement, i.e.

(i) **the continuity of the displacement vector  $\vec{v}_h$  at the vertices  $a$  of  $\mathcal{T}_h$**

$$(29) \quad \vec{v}_h(a^+) = \vec{v}_h(a^-), \quad \forall a^+ = a^- = a \text{ vertex of } \mathcal{T}_h;$$

(ii) **the continuity of the rotation vector  $\vec{\beta}_h$  at the vertices  $a$  of  $\mathcal{T}_h$ ,**

$$(30) \quad \vec{\beta}_h(a^+) = \vec{\beta}_h(a^-), \quad \forall a^+ = a^- = a \text{ vertex of } \mathcal{T}_h,$$

where the vector  $\vec{\beta}_h$  has now three components, i.e.  $\vec{\beta}_h|_K = (\tilde{\beta}_{hi}\vec{a}_h^i)|_K$ ,  $\forall K \in \Omega_h$ . We define the discrete space  $\vec{W}_h$  as

$$\begin{aligned} \vec{W}_h &= \{(\vec{v}_h, \vec{\beta}_h) \in (H^1(\Omega_h))^6; (\vec{v}_h, \vec{\beta}_h)|_K \in (P_1(K))^6, \forall K \in \Omega_h; \\ &\quad (\vec{v}_h, \vec{\beta}_h) \text{ satisfies (29) and (30); } \vec{v}_h(a) = \vec{\beta}_h(a) = \vec{0}, \forall a \in \Gamma_o\}, \end{aligned}$$

provided with the discrete norm

$$\|(\vec{v}_h, \vec{\beta}_h)\|_h = \sum_{K \in \Omega_h} (\|\vec{v}_h\|_{1,K}^2 + \|\vec{\beta}_h\|_{1,K}^2)^{1/2}.$$

Once more, one can prove that the relations

$$(31) \quad \begin{cases} \vec{v}_h(a) = \vec{v}_h(a), \\ \vec{\beta}_h(a) = \vec{\beta}_h(a), \end{cases} \quad \forall a \text{ vertex of } \mathcal{T}_h$$

define a bijection  $\mathcal{G}_h$  between  $\vec{W}_h$  and

$$\vec{W}_h = \{(\vec{v}_h, \vec{\beta}_h) \in (H^1(\Omega_h))^6; (\vec{v}_h, \vec{\beta}_h)|_K \in (P_1(K))^6, \forall K \in \Omega_h; \vec{v}_h|_{\Gamma_o} = \vec{\beta}_h|_{\Gamma_o} = \vec{0}\}.$$

The functions  $(\vec{v}_h, \vec{\beta}_h)$  in  $\vec{W}_h$  are completely determined by the data of the  $6N_h$  degrees of freedom  $\{(\vec{v}_h(a), \vec{\beta}_h(a)), \text{ a vertex of } \mathcal{T}_h\}$  ( $N_h$  being the number of vertices  $a \in \mathcal{T}_h$  such that

$a \notin \Gamma_o$ ).

The discrete problem associated with the faceted middle surface  $\mathcal{S}_h$  is : find  $(\vec{u}_h, \vec{\delta}_h) \in \vec{W}_h$ , such that

$$(32) \quad \tilde{a}_h[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] = \tilde{\ell}_h(\vec{v}_h), \quad \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h,$$

with

$$\begin{aligned} \tilde{a}_h[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] &= \int_{\Omega_h} \{ e E_h^{\alpha\beta\lambda\mu} [\tilde{\gamma}_{h\alpha\beta}(\vec{u}_h) \tilde{\gamma}_{h\lambda\mu}(\vec{v}_h) + \frac{e^2}{12} \tilde{\gamma}_{h\alpha\beta}(\vec{\delta}_h) \tilde{\gamma}_{h\lambda\mu}(\vec{\beta}_h)] \\ &\quad + \frac{5}{6} e E_h^{\alpha\beta\gamma} \tilde{\gamma}_{h\alpha\beta}(\vec{u}_h, \vec{\delta}_h) \tilde{\gamma}_{h\gamma\beta}(\vec{v}_h, \vec{\beta}_h) \} \sqrt{a_h} d\xi + k (\vec{\delta}_{h3}, \vec{\beta}_{h3})_{1, \Omega_h}, \end{aligned}$$

will  $k$  being a positive constant.

By using the bijection  $\mathcal{G}_h$ , we associate to (32) the discrete problem : find  $(\vec{u}_h, \vec{\delta}_h) \in \vec{W}_h$  such that :

$$(33) \quad \hat{a}_h[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] = \hat{\ell}_h(\vec{v}_h), \quad \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h,$$

with the correspondences

$$\begin{aligned} \hat{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)] &= \tilde{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)], \quad \forall (\vec{v}_h, \vec{\beta}_h) = \mathcal{G}_h(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h, \\ &\quad \forall (\vec{w}_h, \vec{\omega}_h) = \mathcal{G}_h(\vec{w}_h, \vec{\omega}_h), (\vec{w}_h, \vec{\omega}_h) \in \vec{W}_h, \end{aligned}$$

$$\hat{\ell}_h(\vec{v}_h) = \tilde{\ell}_h(\vec{v}_h), \quad \forall \vec{v}_h = \mathcal{G}_h(\vec{v}_h), \quad \vec{v}_h \in \vec{W}_h.$$

Again following [Bernadou M., Trouné P., Ducatel Y. (1989)], we have :

$$(34) \quad | \gamma_{\alpha\beta}(\vec{v}_h) - \tilde{\gamma}_{h\alpha\beta}(\vec{v}_h) |_{0,K} \leq Ch \| \vec{v}_h \|_{1,K}, \quad \forall \vec{v}_h = \mathcal{G}_h(\vec{v}_h), \vec{v}_h \in \vec{W}_h,$$

and we easily derive :

$$(35) \quad \left\{ \begin{array}{l} | \gamma_{\alpha\beta}(\vec{v}_h, \vec{\beta}_h) - \tilde{\gamma}_{h\alpha\beta}(\vec{v}_h, \vec{\beta}_h) |_{0,K} \leq Ch (\| \vec{v}_h \|_{1,K} + \| \vec{\beta}_h \|_{1,K}), \\ \forall (\vec{v}_h, \vec{\beta}_h) = \mathcal{G}_h(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h. \end{array} \right.$$

But we only get :

$$(36) \quad \left\{ \begin{array}{l} | \kappa_{\alpha\beta}(\vec{\beta}_h) - \tilde{\gamma}_{h\alpha\beta}(\vec{\beta}_h) |_{0,K} \leq C(h \| \vec{\beta}_h \|_{1,K} + | b_{\alpha\beta} |_{0,\infty,K} \| \beta_{h3} \|_{0,K}), \\ \forall \vec{\beta}_h = \mathcal{G}_h(\vec{\beta}_h), \vec{\beta}_h \in \vec{W}_h. \end{array} \right.$$

Therefore, we find as a first consistency error estimate

$$(37) \quad \left\{ \begin{array}{l} | a[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)] + k(\beta_{h3}, \omega_{h3})_{1,\Omega} - \tilde{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)] | \\ \leq C \{ h (\| \vec{v}_h \|_{1,\Omega}^2 + \| \vec{\beta}_h \|_{1,\Omega}^2)^{1/2} (\| \vec{w}_h \|_{1,\Omega}^2 + \| \vec{\omega}_h \|_{1,\Omega}^2)^{1/2} \\ \quad + \varepsilon \| \beta_{h3} \|_{0,\Omega} \| \omega_{h3} \|_{0,\Omega} \}, \\ \forall (\vec{v}_h, \vec{\beta}_h) = \mathcal{G}_h(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h, \quad \forall (\vec{w}_h, \vec{\omega}_h) = \mathcal{G}_h(\vec{w}_h, \vec{\omega}_h), (\vec{w}_h, \vec{\omega}_h) \in \vec{W}_h, \end{array} \right.$$

where  $\varepsilon = \sup_{\Omega_h} |eb_{\alpha\beta}|$ . The second consistency error estimate is

$$(38) \quad |\ell(\vec{v}_h) - \tilde{\ell}_h(\vec{v}_h)| \leq Ch \|\vec{p}\|_{0,\Omega} \|\vec{v}_h\|_{1,\Omega}, \quad \forall \vec{v}_h = \mathcal{G}_h(\vec{v}_h), \quad \vec{v}_h \in \vec{W}_h.$$

In order to apply Strang's second lemma, we prove the uniform  $\vec{W}_h$ -ellipticity of the discrete bilinear form  $\hat{a}_h[.,.]$  (which also give the existence and uniqueness of a solution to the discrete problems (32) and (33)) :

$$\begin{aligned} \hat{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h)] &= a[(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h)] + k \|\beta_{h3}\|_{1,\Omega}^2 \\ &\quad + \{\hat{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h)] - a[(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h)] - k \|\beta_{h3}\|_{1,\Omega}^2\} \end{aligned}$$

and from (33) it follows that there exists  $\alpha$  and  $C$  positive constants (independent of  $h$ ) such that :

$$\hat{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h)] \geq (\alpha - Ch)(\|\vec{v}_h\|_{1,\Omega}^2 + \|\vec{\beta}_h\|_{1,\Omega}^2) + (k - C\varepsilon) \|\beta_{h3}\|_{1,\Omega}^2$$

which establishes, for  $h$  sufficiently small and  $k$  sufficiently large (as  $\varepsilon$  is supposed to be small compared to unity), that there exists a positive constant  $\hat{\alpha}$  independent of  $h$  such that :

$$(39) \quad \hat{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h)] \geq \hat{\alpha}(\|\vec{v}_h\|_{1,\Omega}^2 + \|\vec{\beta}_h\|_{1,\Omega}^2), \quad \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h.$$

Thus, we obtain the following convergence theorem :

**Theorem 3.1** : Assume that the continuous problem (27) admits a unique solution  $(\vec{u}, \vec{\delta}) \in \vec{W} \cap (H^2(\Omega))^5$  for any  $\vec{p} \in (L^2(\Omega))^3$ . Then, for sufficiently small  $h$ , and for  $k$  large enough, there exists a constant  $C > 0$ , independent of  $h$ , such that :

$$(40) \quad \|\vec{u} - \vec{u}_h\|_{1,\Omega} + \|\vec{\delta} - \vec{\delta}_h\|_{1,\Omega} \leq Ch(\|\vec{u}\|_{2,\Omega} + \|\vec{\delta}\|_{2,\Omega} + \|\vec{p}\|_{0,\Omega}).$$

■

**Proof** : By noticing that  $(\vec{u}, \vec{\delta})$  with  $\delta_3 = 0$  is the solution of the continuous problem

$$a[(\vec{u}, \vec{\delta}), (\vec{v}, \vec{\beta})] + k(\delta_3, \beta_3)_{1,\Omega} = \ell(\vec{v}), \quad \forall (\vec{v}, \vec{\beta}, \beta_3) \in \vec{W} \times \{\beta \in H^1(\Omega), \beta|_{\Gamma_o} = 0\}$$

so its  $\vec{W}$ -interpolate verifies  $\pi_h \delta_3 \equiv 0$ , one gets from interpolation theory :

$$\inf_{(\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h} (\|\vec{u} - \vec{v}_h\|_{1,\Omega}^2 + \|\vec{\delta} - \vec{\beta}_h\|_{2,\Omega}^2)^{1/2} \leq Ch(\|\vec{u}\|_{2,\Omega} + \|\vec{\delta}\|_{2,\Omega}),$$

and from (37) :

$$\left\{ \begin{aligned} &\inf_{(\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h} \sup_{(\vec{w}_h, \vec{\omega}_h) \in \vec{W}_h} \frac{|a[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)] + k(\beta_{h3}, \omega_{h3})_{0,\Omega} - \hat{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)]|}{(\|\vec{w}_h\|_{1,\Omega}^2 + \|\vec{\omega}_h\|_{1,\Omega}^2)^{1/2}} \\ &\leq Ch(\|\vec{v}\|_{1,\Omega} + \|\vec{\delta}\|_{1,\Omega}) \end{aligned} \right.$$

(where we have used  $\|\pi_h v\|_{1,\Omega} \leq C' \|v\|_{1,\Omega}$ ). And, from (38), one gets

$$\sup_{(\vec{w}_h, \vec{\omega}_h) \in \vec{W}_h} \frac{|\ell(\vec{w}_h) - \hat{\ell}_h(\vec{w}_h)|}{(\|\vec{w}_h\|_{1,\Omega}^2 + \|\vec{\omega}_h\|_{1,\Omega}^2)^{1/2}} \leq Ch \|\vec{p}\|_{0,\Omega}.$$

Then, a classical application of Strang's lemma (using (39)) combined with the estimates above leads to (40). ■

### 3.3 A constant curvature finite element method for deep moderately thick shells

In this paragraph, we give an approximation of [Naghdi P.M. (1972)] model, i.e. find  $(\vec{u}, \vec{\delta}) \in \vec{W}$ , such that

$$(41) \quad a[(\vec{u}, \vec{\delta}), (\vec{v}, \vec{\beta})] = \ell(\vec{v}), \quad \forall (\vec{v}, \vec{\beta}) \in \vec{W},$$

now with

$$\begin{aligned} a[(\vec{u}, \vec{\delta}), (\vec{v}, \vec{\beta})] &= \int_{\Omega} \left\{ e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{u}) \gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12} \chi_{\alpha\beta}(\vec{u}, \vec{\delta}) \chi_{\lambda\mu}(\vec{v}, \vec{\beta})] \right. \\ &\quad \left. + \frac{5}{6} e E^{\alpha\beta\gamma 3} \gamma_{\alpha\beta}(\vec{u}, \vec{\delta}) \gamma_{\gamma 3}(\vec{v}, \vec{\beta}) \right\} \sqrt{a} \, d\xi. \end{aligned}$$

Following [Coutris N. (1978)], this continuous problem admits a unique solution. We will use the same finite element spaces as in the previous paragraph. It is clear from the error estimate (36) that if we want to approximate consistently on each facet the change of curvature tensor

$$\chi_{\alpha\beta}(\vec{v}, \vec{\beta}) = \frac{1}{2} (\beta_{\alpha|\beta} + \beta_{\beta|\alpha}) - \frac{1}{2} (b_{\beta}^{\lambda} v_{\lambda|\alpha} + b_{\alpha}^{\lambda} v_{\lambda|\beta}) + c_{\alpha\beta} v_3,$$

we will have to use some approximations of the curvatures (as it was done in paragraph 3.1). According to one of the methods described in remark 3.1, we will denote  $b_{h\alpha\beta}$  such an approximation and  $b_{h\alpha}^{\lambda} = a_h^{\lambda\beta} b_{h\alpha\beta}$ . In the present case, we choose to introduce the sixth degree of freedom in a more **natural** way, avoiding the condition on the arbitrary coefficient  $k$  with regard to  $\varepsilon (\sim \frac{\varepsilon}{R})$ . Now, on each triangle, we associate to the displacement  $\vec{v}_h$  and rotation  $\vec{\beta}_h$  fields the approximations  $\tilde{\gamma}_{h\alpha\beta}(\vec{v}_h)$ ,  $\tilde{\gamma}_{h\alpha 3}(\vec{v}_h, \vec{\beta}_h)$ , and

$$(42) \quad \tilde{\chi}_{h\alpha\beta}(\vec{v}_h, \vec{\beta}_h) = \frac{1}{2} (\tilde{\beta}_{h\alpha,\beta} + \tilde{\beta}_{h\beta,\alpha}) - \frac{1}{2} (b_{h\beta}^{\lambda} \tilde{v}_{h\lambda,\alpha} + b_{h\alpha}^{\lambda} \tilde{v}_{h\lambda,\beta}) + b_{h\alpha\beta} \tilde{\beta}_{h3}.$$

Of course, we set the compatibility relations (29) and (30) at the vertices of  $\mathcal{T}_h$ . And the relations (31) give the existence of the bijection  $\mathcal{G}_h$  between the discrete spaces  $\vec{W}_h$  and  $\vec{W}_h$ .

Next the discrete problem associated with the faceted middle surface  $\mathcal{S}_h$  is : find  $(\vec{u}_h, \vec{\delta}_h) \in \vec{W}_h$ , such that

$$(43) \quad \tilde{a}_h[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] = \tilde{\ell}_h(\vec{v}_h), \quad \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h,$$

with

$$\begin{aligned} \tilde{a}_h[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] &= \int_{\Omega_h} \{ e E_h^{\alpha\beta\lambda\mu} [\tilde{\gamma}_{h\alpha\beta}(\vec{u}_h) \tilde{\gamma}_{h\lambda\mu}(\vec{v}_h) + \frac{e^2}{12} \tilde{\chi}_{h\alpha\beta}(\vec{u}_h, \vec{\delta}_h) \tilde{\chi}_{h\lambda\mu}(\vec{v}_h, \vec{\beta}_h)] \} \\ &\quad + \frac{5}{6} e E_h^{\alpha\beta\gamma\delta} \tilde{\gamma}_{h\alpha\beta}(\vec{u}_h, \vec{\delta}_h) \tilde{\gamma}_{h\gamma\delta}(\vec{v}_h, \vec{\beta}_h) \} \sqrt{a} d\xi + k (\delta_{h3}, \beta_{h3})_{1, \Omega_h}. \end{aligned}$$

and we can define through the bijection  $\mathcal{G}_h$  an equivalent discrete problem to (43), set on the continuous middle surface  $\mathcal{S}$  (see (33)). The error estimates (34), (35) and (38) still apply here, but now we have in addition

$$(44) \quad \left\{ \begin{array}{l} | \chi_{\alpha\beta}(\vec{v}_h, \vec{\beta}_h) - \tilde{\chi}_{h\alpha\beta}(\vec{v}_h, \vec{\beta}_h) |_{0,K} \leq Ch(\| \vec{v}_h \|_{1,K} + \| \vec{\beta}_h \|_{1,K}), \\ \forall (\vec{v}_h, \vec{\beta}_h) = \mathcal{G}_h(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h. \end{array} \right.$$

Thus, we obtain for the first consistency error estimate

$$(45) \quad \left\{ \begin{array}{l} | a[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)] + k(\beta_{h3}, \omega_{h3})_{1, \Omega} - \tilde{a}_h[(\vec{v}_h, \vec{\beta}_h), (\vec{w}_h, \vec{\omega}_h)] | \\ \leq Ch(\| \vec{v}_h \|_{1, \Omega}^2 + \| \vec{\beta}_h \|_{1, \Omega}^2)^{1/2} (\| \vec{w}_h \|_{1, \Omega}^2 + \| \vec{\omega}_h \|_{1, \Omega}^2)^{1/2}, \\ \forall (\vec{v}_h, \vec{\beta}_h) = \mathcal{G}_h(\vec{v}_h, \vec{\beta}_h), (\vec{v}_h, \vec{\beta}_h) \in \vec{W}_h, \forall (\vec{w}_h, \vec{\omega}_h) = \mathcal{G}_h(\vec{w}_h, \vec{\omega}_h), (\vec{w}_h, \vec{\omega}_h) \in \vec{W}_h. \end{array} \right.$$

Then, using similar arguments as in the previous section, one can establish :

**Theorem 3.2 :** *Assume that the solution of the continuous problem (41) belongs to the space  $(H^2(\Omega))^5$  for any  $\vec{p} \in (L^2(\Omega))^3$ . Then, for sufficiently small  $h$ , there exists a constant  $C > 0$ , independent of  $h$ , such that (40) holds.*

■

It seems that this result is still available if we only perturb the discrete bilinear form with  $L^2(\Omega_h)$ -product of third components of the rotation vector, but this needs a more precise (and much more detailed) analysis. Of course, the constant  $k$  should not be taken too large (as a constant  $C$  is proportional to  $k$ ).

### 3.4 A discrete Kirchhoff facet finite element method for deep thin shells

The former methods, when applied in the thin shell limit, is known to **lock**, and this motivates the use of discrete Kirchhoff finite element methods (in this sense, see [Pitkaranta J. (1988)] work). In this paragraph, we choose to study the consistency of a flat plate DKT approximation of thin shells (i.e. without any approximation of shear stresses). We will first analyze a simpler scheme than the one studied in [Bernadou M., Mato Eiroa P., Trouvé P. (1990)] — various reasons leads us to recommand a curved facet approximation of the geometry, as in [Bernadou M., Trouvé P. (1990)] with this DKT finite element for thin shells. Secondly

we will take into account the effect of the discretization of the geometry as in the previous section (  $\vec{\phi}_h$  and  $b_{h\alpha\beta}$  ).

The  $C^0(\Omega)$  -DKT finite element method associated with the continuous middle surface  $\mathcal{S}$  is characterized by the following discrete space

$$\begin{aligned}\vec{Z}_h = \{(\vec{v}_h, \vec{\beta}_h) \in (H^1(\Omega))^5 ; (\vec{v}_h, \vec{\beta}_h) \in (P_1(K))^2 \times P'_3(K) \times (P_1(K))^2, \text{ and} \\ \beta_{h\alpha}(a_i) = -v_{h3,\alpha}(a_i) - b_\alpha^\lambda(a_i)v_{h\lambda}(a_i), a_i \text{ vertex of } K, i = 1, 2, 3, \\ \alpha = 1, 2, \forall K \in \Omega_h ; \vec{v}_h|_{\Gamma_o} = \vec{\beta}_h|_{\Gamma_o} = \vec{0}\},\end{aligned}$$

and the discrete problem is : find  $(\vec{u}_h, \vec{\delta}_h) \in \vec{Z}_h$ , such that

$$(46) \quad a[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] = \ell(\vec{v}_h), \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{Z}_h,$$

where

$$a[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] = \int_{\Omega} e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{u}_h) \gamma_{\lambda\mu}(\vec{v}_h) + \frac{e^2}{12} \chi_{\alpha\beta}(\vec{u}_h, \vec{\delta}_h) \chi_{\lambda\mu}(\vec{v}_h, \vec{\beta}_h)] \sqrt{a} d\xi.$$

By eliminating the  $\vec{\beta}_h$ -unknowns, we can reformulate it as an equivalent problem set in the discrete space

$$\vec{V}_h = \{\vec{v}_h \in (H^1(\Omega)) \times H^2(\Omega_h) ; \vec{v}_h|_K \in (P_1(K))^2 \times P'_3(K), \forall K \in \Omega_h ; \vec{v}_h|_{\Gamma_o} = \partial_n v_{h3}|_{\Gamma_o} = 0\}$$

i.e. find  $\vec{u}_h \in \vec{V}_h$  such that

$$(47) \quad d_h(\vec{u}_h, \vec{v}_h) = \ell(\vec{v}_h), \forall \vec{v}_h \in \vec{V}_h,$$

where we have denoted

$$d_h(\vec{u}_h, \vec{v}_h) = \int_{\Omega_h} e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{u}_h) \gamma_{\lambda\mu}(\vec{v}_h) + \frac{e^2}{12} \bar{\rho}_{h\alpha\beta}(\vec{u}_h) \bar{\rho}_{h\lambda\mu}(\vec{v}_h)] \sqrt{a} d\xi,$$

and

$$(48) \quad \bar{\rho}_{h\alpha\beta}(\vec{v}_h) = -\chi_{\alpha\beta}(\vec{v}_h, \vec{\beta}_h), \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{Z}_h,$$

or again

$$(49) \quad \begin{cases} \bar{\rho}_{h\alpha\beta}(\vec{v}_h) = \frac{1}{2} [(\partial_{h\alpha}(\vec{v}_h))_{,\beta} + (\partial_{h\beta}(\vec{v}_h))_{,\alpha}] - \Gamma_{\alpha\beta}^\lambda \partial_{h\lambda}(\vec{v}_h) - c_{\alpha\beta} v_{h3} \\ \quad + b_\alpha^\lambda v_{h\lambda|\beta} + b_\beta^\lambda v_{h\lambda|\alpha} + b_{\alpha|\beta}^\lambda v_{h\lambda}, \end{cases}$$

and

$$(50) \quad \partial_{h\alpha}(\vec{v}_h) = -\beta_{h\alpha} - b_\alpha^\lambda v_{h\lambda}, \alpha = 1, 2, \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{Z}_h.$$

Therefore, we have built a nonconforming approximation of (13) for two reasons : firstly  $\vec{V}_h \not\subset \vec{V}$  (the continuity of the normal derivatives of the transverse component of the displacement through any interface between two adjacent element is not ensured), and secondly, we have  $\bar{\rho}_{h\alpha\beta}(\vec{v}_h)|_K \neq \bar{\rho}_{\alpha\beta}(\vec{v}_h)|_K$ ,  $\forall K \in \mathcal{T}_h$ ,  $\forall \vec{v}_h \in \vec{V}_h$ . The analysis of such a method follows the steps of the work of [Bernadou M., Mato Eiroa P., Trouné P. (1990)]. First of all, we have the basic lemma :

**Lemma 3.1** : *For any  $\vec{v}_h \in \vec{V}_h$ , there exists a constant  $C > 0$ , independent of  $h$ , such that :*

$$(51) \quad \left\{ \begin{array}{l} |v_{h3,\alpha} - \partial_{h\alpha}(\vec{v}_h)|_{m,K} \leq Ch^{1-m}(|v_{h1}|_{1,K}^2 + |v_{h2}|_{1,K}^2 + |v_{h3,\alpha}|_{1,K}^2)^{1/2}, \\ m = 0, 1, \alpha = 1, 2. \end{array} \right.$$

■

**Proof** : From (50), we get on each triangle  $K \in \Omega_h$  :

$$\begin{aligned} \partial_{h\alpha}(\vec{v}_h)(\mu) &= -\beta_{h\alpha}(\mu) - b_\alpha^\lambda(\mu)v_{h\lambda}(\mu) \\ &= -\sum_{i=1}^3 \lambda_i(\mu)[\beta_{h\alpha}(a_i) + b_\alpha^\lambda(\mu)v_{h\lambda}(a_i)] \end{aligned}$$

and by using the "discrete Kirchhoff constraints" :

$$\begin{aligned} \partial_{h\alpha}(\vec{v}_h)(\mu) &= -\sum_{i=1}^3 \lambda_i(\mu)[v_{h3,\alpha}(a_i) + (b_\alpha^\lambda(a_i) - b_\alpha^\lambda(\mu))v_{h\lambda}(a_i)] \\ &= \pi_K^1(v_{h3,\alpha}) + \pi_K^1(b_\alpha^\lambda v_{h\lambda}) - b_\alpha^\lambda(\mu)v_{h\lambda}(\mu) \end{aligned}$$

where  $\pi_K^1$  denotes the  $P_1(K)$ -interpolation operator. Thus, we derive (51) from the estimates

$$|v_{h3,\alpha} - \pi_K^1(v_{h3,\alpha})|_{m,K} \leq Ch^{1-m} |v_{h3,\alpha}|_{1,K}, \quad m = 0, 1, \alpha = 1, 2,$$

$$|b_\alpha^\lambda v_{h\lambda} - \pi_K^1(b_\alpha^\lambda v_{h\lambda})|_{m,K} \leq Ch^{1-m}(|v_{h1}|_{1,K} + |v_{h2}|_{1,K}), \quad m = 0, 1, \alpha = 1, 2,$$

if we use  $b_\alpha^\lambda \in W^{1,\infty}(\Omega)$ .

■

Then, we get the following consistency error estimates :

**Lemma 3.2** : *Assume that the solution  $\vec{u}$  of the continuous problem (19) belongs to  $\vec{V} \cap (H^2(\Omega))^2 \times H^3(\Omega)$ , then there exists a constant  $C$ , independent of  $h$ , such that :*

$$(52) \quad \sup_{\vec{w}_h \in \vec{V}_h} \frac{|\ell(\vec{w}_h) - d_h(\vec{u}, \vec{w}_h)|}{\|\vec{w}_h\|_h} \leq Ch(\|u_1\|_{2,\Omega}^2 + \|u_2\|_{2,\Omega}^2 + \|u_3\|_{3,\Omega}^2)^{1/2}.$$

■

**Proof** : By definition one has

$$\bar{\rho}_{h\alpha\beta}(\vec{u}) = -\chi_{\alpha\beta}(\vec{u}, \vec{\delta}) = \bar{\rho}_{\alpha\beta}(\vec{u}),$$

and using Green's formula, equilibrium equations and boundary conditions, one gets for any  $(\vec{w}_h, \vec{\omega}_h) \in \vec{Z}_h$  (see also [Bernadou M., Mato Eiroa P., Trouvé P. (1990)])

$$\begin{aligned} \ell(\vec{w}_h) - d_h(\vec{u}, \vec{w}_h) &= \int_{\Omega_h} [p^i w_{hi} - \bar{n}^{\alpha\beta}(\vec{u}) \gamma_{\alpha\beta}(\vec{w}_h) - \bar{m}^{\alpha\beta}(\vec{u}) \bar{\rho}_{h\alpha\beta}(\vec{w}_h)] \sqrt{a} \, d\xi \\ &= \int_{\Omega_h} \bar{m}^{\alpha\beta}(\vec{u}) |_{\beta} [w_{h3,\alpha} - \partial_{h\alpha}(\vec{w}_h)] \sqrt{a} \, d\xi \end{aligned}$$

where  $\bar{n}^{\alpha\beta}(\vec{u})$  and  $\bar{m}^{\alpha\beta}(\vec{u})$  are respectively the stress resultant and stress couple tensor. And (52) follows directly from lemma 3.1 (with  $m = 0$ ).  $\blacksquare$

Next, using interpolation error estimates and similar arguments as [Bernadou M., Mato Eiroa P., Trouvé P. (1990)] to obtain the uniform  $\vec{V}_h$ -ellipticity of the discrete bilinear form  $d_h(\cdot, \cdot)$ , which establishes the existence and uniqueness of a solution to problems (46) and (47), we obtain convergence of the method (using Strang's first lemma), i.e.

$$(53) \quad \|\vec{u} - \vec{u}_h\|_h \leq Ch(\|u_1\|_{2,\Omega}^2 + \|u_2\|_{2,\Omega}^2 + \|u_3\|_{3,\Omega}^2)^{1/2}.$$

Similarly to the previous section, we now consider the discrete problem associated with the faceted middle surface  $\mathcal{S}_h$ . On each triangle, we use the discrete Kirchhoff constraints for plates, i.e.

$$(54) \quad \tilde{\beta}_{h\alpha}(a_i) = -\tilde{v}_{h3,\alpha}(a_i), \quad a_i \text{ vertex of } K, \quad i = 1, 2, 3, \quad \alpha = 1, 2, \quad \forall K \in \Omega_h,$$

which can be seen through relations (31) as some  $0(h)$ -approximations of the former constraints associated with the continuous middle surface. Then, taking into account a sixth degree of freedom at each node, we associate to the displacement  $\vec{v}_h$  and rotation  $\vec{\beta}_h$  fields the approximate strains tensor  $\tilde{\gamma}_{h\alpha\beta}(\vec{v}_h)$ ,  $\tilde{\chi}_{h\alpha\beta}(\vec{v}_h, \vec{\beta}_h)$  and  $\tilde{\gamma}_{h\alpha 3}(\vec{v}_h, \vec{\beta}_h)$ , see (17) (28) (42), and using the compatibility relations (29) and (30), we can introduce the discrete space :

$$\begin{aligned} \vec{Z}_h &= \{(\vec{v}_h, \vec{\beta}_h) \in (H^1(\Omega_h))^6; (\vec{v}_h, \vec{\beta}_h) \in (P_1(K))^2 \times P'_3(K) \times (P_1(K))^2, \text{ and} \\ &\quad (\vec{v}_h, \vec{\beta}_h)|_K \text{ satisfies (54), } \forall K \in \Omega_h; \vec{v}_h(a) = \vec{\beta}_h(a) = 0, \forall a \text{ vertex} \in \Gamma_o; \\ &\quad (\vec{v}_h, \vec{\beta}_h) \text{ satisfies (29) and (30)}\}. \end{aligned}$$

Therefore the discrete problem is : find  $(\vec{u}_h, \vec{\delta}_h) \in \vec{Z}_h$ , such that

$$(55) \quad \tilde{a}_h[(\vec{u}_h, \vec{\delta}_h), (\vec{v}_h, \vec{\beta}_h)] = \tilde{\ell}_h(\vec{v}_h), \quad \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{Z}_h,$$

with the same bilinear form  $\tilde{a}_h[\cdot, \cdot]$  as in (43). The numerical analysis of this problem follows the lines of the previous studies, and the key point is certainly the error estimate (44), which implies (45).

An equivalent form of this problem is obtained by eliminating the  $\vec{\beta}_h$ -unknowns on each



triangle, and set the problem in the discrete space

$$\begin{aligned}\vec{Y}_h &= \{(\vec{v}_h, \vec{\beta}_{h3}) \in (H^1(\Omega_h))^2 \times H^2(\Omega_h) \times H^1(\Omega_h); \\ &\quad (\vec{v}_h, \vec{\beta}_{h3})|_K \in (P_1(K))^2 \times P'_3(K) \times P_1(K), \forall K \in \Omega_h; \\ &\quad \vec{v}_h \text{ satisfies (29) and (30) with } \vec{\beta}_h(a) = -\tilde{v}_{h3,\alpha}(a)\vec{a}_h^\alpha + \vec{\beta}_{h3}(a)\vec{a}_h^3, \\ &\quad \forall a \text{ vertex of } \mathcal{T}_h; \vec{v}_h(a) = \vec{\beta}_h(a) = 0, \forall a \text{ vertex of } \Gamma_o\},\end{aligned}$$

i.e. find  $(\vec{u}_h, \vec{\delta}_{h3}) \in \vec{Y}_h$  such that

$$(56) \quad \tilde{d}_h[(\vec{u}_h, \vec{\delta}_{h3}), (\vec{v}_h, \vec{\beta}_{h3})] = \tilde{\ell}_h(\vec{v}_h), \forall (\vec{v}_h, \vec{\beta}_{h3}) \in \vec{Y}_h,$$

where

$$\begin{aligned}\tilde{d}_h[(\vec{u}_h, \vec{\delta}_{h3}), (\vec{v}_h, \vec{\beta}_{h3})] &= \int_{\Omega_h} e E_h^{\alpha\beta\lambda\mu} [\tilde{\gamma}_{h\alpha\beta}(\vec{u}_h) \tilde{\gamma}_{h\lambda\mu}(\vec{v}_h) \\ &\quad + \frac{e^2}{12} \tilde{\rho}_{h\alpha\beta}(\vec{u}_h, \vec{\delta}_h) \tilde{\rho}_{h\lambda\mu}(\vec{v}_h, \vec{\beta}_{h3})] \sqrt{a} d\xi + k(\vec{\delta}_{h3}, \vec{\beta}_{h3})_{1,\Omega_h}\end{aligned}$$

with  $k$  a positive constant and

$$\tilde{\rho}_{h\alpha\beta}(\vec{v}_h, \vec{\beta}_{h3}) = \frac{1}{2} [(\tilde{\partial}_{h\alpha}(\vec{v}_h))_{,\beta} + (\tilde{\partial}_{h\beta}(\vec{v}_h))_{,\alpha}] + \frac{1}{2} (b_{h\beta}^\lambda \tilde{v}_{h\lambda,\alpha} + b_{h\alpha}^\lambda \tilde{v}_{h\lambda,\beta}) - b_{h\alpha\beta} \tilde{\beta}_{h3}$$

and the local discrete derivatives

$$\tilde{\partial}_{h\alpha}(\vec{v}_h) = -\tilde{\beta}_{h\alpha}, \alpha = 1, 2, \forall (\vec{v}_h, \vec{\beta}_h) \in \vec{Z}_h.$$

Then, it is easily shown that relations (31) defines a bijection between the spaces  $\vec{Y}_h$  and  $\vec{V}_h = \vec{V}_h \times \{\beta_h \in H^1(\Omega); \beta_h|_K \in P_1(K), \forall K \in \Omega_h; \beta_h|_{\Gamma_o} = 0\}$  such that we can associate to (56) and equivalent discrete problem : find  $(\vec{u}_h, \hat{\delta}_{h3}) \in \vec{Y}_h$  such that

$$(57) \quad \hat{d}_h[(\vec{u}_h, \hat{\delta}_{h3}), (\vec{v}_h, \beta_{h3})] = \hat{\ell}_h(\vec{v}_h), \forall (\vec{v}_h, \beta_{h3}) \in \vec{Y}_h,$$

where we have used the correspondences

$$\begin{aligned}\hat{d}_h[(\vec{v}_h, \beta_{h3}), (\vec{w}_h, \omega_{h3})] &= \tilde{d}_h[(\vec{v}_h, \vec{\beta}_{h3}), (\vec{w}_h, \vec{\omega}_{h3})], \\ \hat{\ell}_h(\vec{v}_h) &= \tilde{\ell}_h(\vec{v}_h).\end{aligned}$$

The convergence analysis follows the error estimate  $|\bar{\rho}_{\alpha\beta}(\vec{v}_h) - \tilde{\rho}_{h\alpha\beta}(\vec{v}_h, \beta_{h3})|_{0,K}$  as in [Bernadou M., Trouvé P. (1990)], which is derived from (44). Using Strang's lemmas this leads to the discretization error

$$(58) \quad \|\vec{u} - \vec{u}_h\|_h \leq Ch(\|u_1\|_{2,\Omega}^2 + \|u_2\|_{2,\Omega}^2 + \|u_3\|_{3,\Omega}^2)^{1/2}.$$

Let us add that a similar finite element method was proposed by [Bathe K.J., Ho L.W. (1981)] by taking instead of the approximate bending strain tensor  $\tilde{\chi}_{h\alpha\beta}$  (42) the tensor  $\tilde{\kappa}_{h\alpha\beta}$  (28). It is now clear that this method can be justified for **sufficiently shallow** thin shells, but not for general shells.

## 4 CONCLUSION

Besides the great variety of finite element methods presented in this paper for thick/thin shells, the major conclusion is that the approximation of the geometry by triangular plane facets cannot rely only on plate bending strains elementwise to converge. One need to add some approximations of the curvatures in the approximate change of curvature tensor. This may raise the interest of curved facet-like finite element methods in the case of higher-order methods for shells problems (see [Bernadou M., Trouvé P. (1990)]).

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Imprimé en France  
par  
l'Institut National de Recherche en Informatique et en Automatique



ISSN 0249 - 6399